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Supplement to Cosine Methods for Nonlinear Second-Order Hyperbolic Equations

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S2. CONSISTENCY AND PRELIMINARY ERROR ESTIMATES

Proof of Lemma 2.3. For $\varphi \in S_h$, using (1.6) we have

$$(2.9)\quad (W^{n+1}-2W^n+W^{n-1},\varphi)=(W^{n+1}-u^{n+1}-2(W^n-u^n)+W^{n-1}-u^{n-1},\varphi)$$
$$+(u^{n+1}-2u^n+u^{n-1},\varphi)\le ck^2h^r\|\varphi\|+(u^{n+1}-2u^n+u^{n-1},\varphi).$$

Since $L_nW^n=PL(t_n)u^n=f^n-Pu^{(2)n}$ by (1.1), we have

$$(2.10)\quad k^2(q_1L_{n+1}W^{n+1}-2p_1L_nW^n+q_1L_{n-1}W^{n-1})=k^2(q_1f^{n+1}-2p_1f^n+q_1f^{n-1})$$
$$-k^2P(q_1u^{(2)n+1}-2p_1u^{(2)n}+q_1u^{(2)n-1}).$$

From (1.1) we have that $u^{(4)}=-L(-Lu+f)-2L^{(1)}u^{(1)}-L^{(2)}u+f^{(2)}$. Hence

$$(2.11)\quad k^4(q_2L_{n+1}{}^2W^{n+1}-2p_2L_n{}^2W^n+q_2L_{n-1}{}^2W^{n-1})$$
$$=k^4(q_2[L_{n+1}{}^2W^{n+1}-PL(t_{n+1})(L(t_{n+1})u^{n+1}-f(t_{n+1}))]$$
$$-2p_2[L_n{}^2W^n-PL(t_n)(L(t_n)u^n-f(t_n))]$$
$$+q_2[L_{n-1}{}^2W^{n-1}-PL(t_{n-1})(L(t_{n-1})u^{n-1}-f(t_{n-1}))]\}$$
$$+k^4P(q_2u^{(4)n+1}-2p_2u^{(4)n}+q_2u^{(4)n-1})$$
$$+2k^4q_2P(L^{(1)}(t_{n+1})u^{(1)n+1}-2L^{(1)}(t_n)u^{(1)n}+L^{(1)}(t_{n-1})u^{(1)n-1})$$
$$+2(q_1-1/12)k^4PL^{(1)}(t_n)u^{(1)n}$$
$$+k^4q_2P(L^{(2)}(t_{n+1})u^{n+1}-2L^{(2)}(t_n)u^n+L^{(2)}(t_{n-1})u^{n-1})$$
$$+(q_1-1/12)k^4PL^{(2)}(t_n)u^n$$
$$-k^4q_2(f^{(2)n+1}-2f^{(2)n}+f^{(2)n-1})-k^4(q_1-1/12)f^{(2)n}.$$

Now note that by (1.1)

(2.12) $L_n^2\mu^n - PL(t_n)(L(t_n)u^n - f(t_n)) = -L_n(P-P_1(t_n))u^{(2)n} + L_n f^n$.

Since $u^{(2)} \in D_L$, we have by (1.3), (1.5) that

(2.13) $$k^4(L_n(P-P_1(t_n))u^{(2)n},\varphi) = k^4((P-P_1(t_n))u^{(2)n}, L_n\varphi)$$
$$\le ck^4h^r\|L_n\varphi\|, \quad \varphi \in S_h.$$

For the second-order centered difference quotients of the right-hand side of (2.11) we have

(2.14) $$k^4q_2(2[L^{(1)}(t_{n+1})u^{(1)n+1} - 2L^{(1)}(t_n)u^{(1)n} + L^{(1)}(t_{n-1})u^{(1)n-1}]$$
$$+[L^{(2)}(t_{n+1})u^{n+1} - 2L^{(2)}(t_n)u^n + L^{(2)}(t_{n-1})u^{n-1}]$$
$$-[f^{(2)}(t_{n+1}) - 2f^{(2)}(t_n) + f^{(2)}(t_{n-1})],\varphi)$$
$$\le q_2k^6 \sup_{\tau\in[t_{n-1},t_{n+1}]} (2\|D_\tau^2(L^{(1)}(\tau)u^{(1)}(\tau))\|$$
$$+\|D_\tau^2(L^{(2)}(\tau)u(\tau))\| + \|D_\tau^4 f(\tau)\|)\|\varphi\| \le ck^6\|\varphi\|, \quad \varphi \in S_h.$$

Noting finally that the fourth-order accuracy of the cosine methods in hand gives, cf. [3],

(2.15) $$([u^{n+1} - 2u^n + u^{n-1}] - k^2[q_1u^{(2)n+1} - 2p_1u^{(2)n} + q_1u^{(2)n-1}]$$
$$+k^4[q_2u^{(4)n+1} - 2p_2u^{(4)n} + q_2u^{(4)n-1}],\varphi)$$
$$\le ck^6 \sup_{\tau\in[t_{n-1},t_{n+1}]} \|u^{(6)}(\tau)\|\,\|\varphi\| \le ck^6\|\varphi\|, \quad \varphi \in S_h,$$

we conclude that (2.7) and (2.8) follow from (2.9)-(2.15).□

The remainder of this section is devoted to the estimation of the last three sums in the right-hand side of (2.18), cf. the main body of the paper.

LEMMA 2.5. *Suppose that* $1 \le m \le l \le J-1$, *that the hypotheses of Lemma* 2.4 *hold, and in addition that* $\bar{U}^{n+1} \in Y$, $m \le n \le l$, $U^n \in Y$, $m-1 \le n \le l$. *Then, defining* S_n *by* (2.5), *we have*

(2.20) $$\sum_{n=m}^{l}{}' (S_nE^n, E^{n+1} - E^{n-1})$$
$$\le ck \sum_{n=m}^{l}{}' (h^{-1}(|\hat{e}^{n+1}|_\infty + |e^n|_\infty + |e^{n-1}|_\infty)\|E^{n+1} - E^{n-1}\|^2$$
$$+h^{-1}k^2(|\hat{e}^{n+1}|_\infty\|L_{n+1}{}^{1/2}E^{n+1}\|^2 + |e^n|_\infty\|L_n{}^{1/2}E^n\|^2 + |e^{n-1}|_\infty\|L_{n-1}{}^{1/2}E^{n-1}\|^2)$$
$$+h^{-1}(|\hat{e}^{n+1}|_\infty + |e^n|_\infty + |e^{n-1}|_\infty)k^2\|L_{n+1}{}^{1/2}(E^{n+1} - E^{n-1})\|^2$$
$$+h^{-1}k^4(|\hat{e}^{n+1}|_\infty\|L_{n+1}E^{n+1}\|^2 + |e^n|_\infty\|L_nE^n\|^2 + |e^{n-1}|_\infty\|L_{n-1}E^{n-1}\|^2)$$
$$+h^{-1}k^4(|\hat{e}^{n+1}|_\infty + |e^n|_\infty + |e^{n-1}|_\infty)\|L_{n+1}(E^{n+1} - E^{n-1})\|^2$$
$$+k^3h^{-2}(|\hat{e}^{n+1}|_\infty^2\|L_{n+1}{}^{1/2}E^{n+1}\|^2 + |e^n|_\infty^2\|L_n{}^{1/2}E^n\|^2 + |e^{n-1}|_\infty^2\|L_{n-1}{}^{1/2}E^{n-1}\|^2)$$
$$+k^3h^{-2}(|\hat{e}^{n+1}|_\infty^2 + |e^n|_\infty^2 + |e^{n-1}|_\infty^2)\|L_n{}^{1/2}(E^{n+1} - E^{n-1})\|^2).$$

Proof. Since, by (2.5),

$$\sum_{n=m}^{l}{}' (S_nE^n, E^{n+1} - E^{n-1})$$
$$= \sum_{n=m}^{l}{}' ((Q_{n+1} - \hat{A}_{n+1})E^{n+1} - 2(P_n - B_n)E^n + (Q_{n-1} - A_{n-1})E^n, E^{n+1} - E^{n-1}),$$

the result follows by estimates analogous to those used in the derivation of (2.3) and by the comparability of the norms $\|L_i{}^{1/2}\cdot\|$, $\|L_j{}^{1/2}\cdot\|$ and $\|L_i\cdot\|$, $\|L_j\cdot\|$, which follows from (iii), (1.2) and the agm inequality.□

Using summation by parts, provided $l \ge m+2$, we now write

(2.21) $$\sum_{n=m}^{l}{}' (S_n\mu^n, E^{n+1} - E^{n-1}) = (S_l\mu^l, E^{l+1}) + (S_{l-1}\mu^{l-1}, E^l)$$
$$-(S_m\mu^m, E^{m-1}) - (S_{m+1}\mu^{m+1}, E^m) - \sum_{n=m+1}^{l-1} (S_{n+1}\mu^{n+1} - S_{n-1}\mu^{n-1}, E^n)$$

and estimate the right-hand side in the following three lemmata.

LEMMA 2.6. _Suppose that_ $\hat{U}^{l+1}, \hat{U}^{l}, U^{l}, U^{l-1}, U^{l-2}$ _for_ $i=l$ _and_ $i=m+1$ _exist in_ $S_h \cap Y$ _and that_ U^{l+1}, U^{m+2} _exist in_ S_h. _Moreover, assume that_ (1.4) _and_ (1.9, $j=0$) _hold and that there exists_ $a>0$ _such that_ $kh^{-1} \le a$. _Then for any_ $\epsilon_1, \epsilon_2 > 0$ _there exists a constant_ $C(\epsilon_1, \epsilon_2) > 0$ _such that_

$$(2.22)\quad |(S_l \mu^l, E^{l+1}) + (S_{l-1} \mu^{l-1}, E^l)|$$
$$\le C(\epsilon_1,\epsilon_2) k^2 [\textstyle\sum_{j=1}^{l-1} \|\hat{e}^j\|^2 (1+|\hat{e}^j|_\infty^2) + \sum_{j=l-2}^{l} \|e^j\|^2 (1+|e^j|_\infty^2)]$$
$$+\epsilon_1 k^2 (\|L_{l+1}^{1/2}(E^{l+1}+E^l)\|^2 + \|L_{l+1}^{1/2}(E^{l+1}-E^l)\|^2)$$
$$+\epsilon_2 k^4 (\|L_{l+1}(E^{l+1}+E^l)\|^2 + \|L_{l+1}(E^{l+1}-E^l)\|^2),$$

$$(2.23)\quad |(S_m \mu^m, E^{m-1}) + (S_{m+1} \mu^{m+1}, E^m)| \le \eta_m^{(2)},$$

where

$$(2.24)\quad \eta_m^{(2)} = ck^2 [\textstyle\sum_{j=m+1}^{m+2} \|\hat{e}^j\|^2 (1+|\hat{e}^j|_\infty^2) + \sum_{j=m-1}^{m+1} \|e^j\|^2 (1+|e^j|_\infty^2)]$$
$$+k^2 (\|L_m^{1/2}(E^m+E^{m-1})\|^2 + \|L_m^{1/2}(E^m-E^{m-1})\|^2)$$
$$+k^4 (\|L_m(E^m+E^{m-1})\|^2 + \|L_m(E^m-E^{m-1})\|^2).$$

Proof. We first note that for $0 \le j \le J$, $g \in Y$, $\varphi \in S_h$, we obtain by (v) and (1.9), $|([L_j - L_j(g)]\mu^j, \varphi)| \le c\|u^j - g\| \, \|L_j^{1/2}\varphi\|$. In addition, by (v.d), (2.1) and (1.9),

$$([L_j^2 - L_j^2(g)]\mu^j, \varphi) = ([L_j - L_j(g)]\mu^j, L_j\varphi) + ([L_j(g) - L_j]\mu^j, [L_j - L_j(g)]\varphi)$$
$$+([L_j - L_j(g)]L_j\mu^j, \varphi)$$
$$\le ch^{-1}\|u^j - g\| \, \|\mu^j\|_{1,\infty} \, \|L_j\varphi\| + \|(L_j(g) - L_j)\mu^j\| \, \|(L_j - L_j(g))\varphi\|$$
$$+c\|u^j - g\| \, \|L_j\mu^j\|_{1,\infty} \, \|L_j^{1/2}\varphi\|$$
$$\le ch^{-1}\|u^j - g\| \, \|L_j\varphi\| + c\|u^j - g\| \, \|PL(t_j)u^j\|_{1,\infty} \, \|L_j^{1/2}\varphi\|$$
$$+ch^{-2}\|u^j - g\| \, |u^j - g|_\infty \, \|L_j^{1/2}\varphi\|.$$

Note that (1.4) gives, since $kh^{-1} \le a$, that $k^2\|PL(t_j)u^j\|_{1,\infty} \le c$. Hence, it follows from our hypotheses, (2.5) and the above, that

$$|(S_l\mu^l, E^{l+1})| \le ck^2(\|\hat{e}^{l+1}\| + \|e^l\| + \|e^{l-1}\|)\|L_{l+1}^{1/2}E^{l+1}\|$$
$$+ck^3(\|\hat{e}^{l+1}\| + \|e^l\| + \|e^{l-1}\|) \, \|L_{l+1}E^{l+1}\|$$
$$+ck^2(\|\hat{e}^{l+1}\| \, |\hat{e}^{l+1}|_\infty + \|e^l\| \, |e^l|_\infty + \|e^{l-1}\| \, |e^{l-1}|_\infty)\|L_{l+1}^{1/2}E^{l+1}\|.$$

An entirely analogous bound holds for $(S_{l-1}\mu^{l-1}, E^l)$ and (2.22) is easily deduced; (2.23) and (2.24) also follow easily.□

To treat the summation term in the right-hand side of (2.21), note that for $l \ge m+2$, $m+1 \le n \le l-1$, $m \ge 1$, $l \le J-1$,

$$(2.25)\qquad S_{n+1}\mu^{n+1} - S_{n-1}\mu^{n-1} = \Pi_n^{(1)} + \Pi_n^{(2)},$$

where for $j=1,2$, $m+1 \le n \le l-1$,

$$(2.26)\quad \Pi_n^{(j)} = q_j k^{2j}[(L_{n+2}^j - L_{n+2}^j(\hat{U}^{n+2}))\mu^{n+2} - (L_n^j - L_n^j(\hat{U}^n))\mu^n]$$
$$-2p_j k^{2j}[(L_{n+1}^j - L_{n+1}^j(U^{n+1}))\mu^{n+1} - (L_{n-1}^j - L_{n-1}^j(U^{n-1}))\mu^{n-1}]$$
$$+q_j k^{2j}[(L_n^j - L_n^j(U^n))\mu^n - (L_{n-2}^j - L_{n-2}^j(U^{n-2}))\mu^{n-2}].$$

We estimate first the term $\Pi_n^{(1)}$, which is linear in k^2.

LEMMA 2.7 *Let* $m\ge 1$, $m+2\le l\le J-1$, *and suppose that* $\hat{U}^i$, $m+1\le i\le l+1$, *and* U^i, $m-1\le i\le l$, *belong to* $S_h\cap V$ *and that* (1.9, $j=0,1$) *holds*. *Then*

$$(2.27)\quad \Big|\sum_{n=m-1}^{l+1}(\Pi_n^{(1)},E^n)\Big|\le ck\sum_{n=m+1}^{l+1}\{k^2(\|\hat{e}^n\|^2+\|e^{n-1}\|^2+\|e^{n-2}\|^2)$$
$$+\|\hat{e}^{n+2}-\hat{e}^n\|^2(1+|\hat{e}^n|_\infty^2)+k^2h^{2r}(|e^{n-1}|_\infty^2+|e^{n-2}|_\infty^2)$$
$$+(1+|e^{n-1}|_\infty^2)\|E^{n+1}-E^{n-1}\|^2+(1+|e^{n-2}|_\infty^2)\|E^n-E^{n-2}\|^2$$
$$+k^2\|L_n^{1/2}E^n\|^2\}+ck^2h^{2r}(l-m-1)k.$$

Proof. For $1\le i\le J-1$, $g^{l\pm 1}\in V$, consider the identity

$$(2.28)\quad (L_{l+1}-L_{l+1}(g^{l+1}))\mu^{l+1}-(L_{l-1}-L_{l-1}(g^{l-1}))\mu^{l-1}=[(L_{l+1}-L_{l+1}(g^{l+1}))$$
$$-(L_{l-1}-L_{l-1}(g^{l-1}))]\mu^{l+1}+(L_{l-1}-L_{l-1}(g^{l-1}))(\mu^{l+1}-\mu^{l-1}).$$

Now for $\varphi\in S_h$, by (v.d), (1.9, $j=1$),

$$(2.29)\quad |((L_{l-1}-L_{l-1}(g^{l-1}))(\mu^{l+1}-\mu^{l-1}),\varphi)|$$
$$\le c\|u^{l-1}-g^{l-1}\|\ k\ \sup_{t_{l-1}\le\tau\le t_{l+1}}\|\mu^{(1)}(\tau)\|_{1,\infty}\ \|L_l^{1/2}\varphi\|$$
$$\le ck\|u^{l-1}-g^{l-1}\|\ \|L_l^{1/2}\varphi\|.$$

In addition, by (1.9, $j=0$) and (v.e) we have

$$(2.30)\quad |((L_{l+1}-L_{l+1}(g^{l+1})-L_{l-1}+L_{l-1}(g^{l-1}))\mu^{l+1},\varphi)|$$
$$\le c[\|u^{l+1}-g^{l+1}-u^{l-1}+g^{l-1}\|(1+|u^{l-1}-g^{l-1}|_\infty)$$
$$+k\|u^{l-1}-g^{l-1}\|]\|L_l^{1/2}\varphi\|.$$

There follows then for $g^i=\hat{U}^i$, $i=n+1$, $m+1\le n\le l-1$, that

$$(2.31)\quad k^2|((L_{n+2}-L_{n+2}(\hat{U}^{n+2}))\mu^{n+2}-(L_n-L_n(\hat{U}^n))\mu^n,E^n)|$$
$$\le c[k^3\|\hat{e}^n\|+k^2\|\hat{e}^{n+2}-\hat{e}^n\|(1+|\hat{e}^n|_\infty)]\|L_n^{1/2}E^n\|.$$

Also, for $g^i=U^i$, $i=n$, in (2.29), (2.30), we have for $m+1\le n\le l-1$, by (1.6),

$$(2.32)\quad k^2|((L_{n+1}-L_{n+1}(U^{n+1}))\mu^{n+1}-(L_{n-1}-L_{n-1}(U^{n-1}))\mu^{n-1},E^n)|$$
$$\le c(k^3\|e^{n-1}\|+k^2(\|(u^{n+1}-\mu^{n+1})-(u^{n-1}-\mu^{n-1})\|$$
$$+\|E^{n+1}-E^{n-1}\|)(1+|e^{n-1}|_\infty))\|L_n^{1/2}E^n\|$$
$$\le c[k^3\|e^{n-1}\|+(k^3h^r+k^2\|E^{n+1}-E^{n-1}\|)(1+|e^{n-1}|_\infty)\|L_n^{1/2}E^n\|.$$

An entirely analogous estimate - with $g^i=U^i$, $i=n-1$ in (2.29), (2.30) - and (2.26), (2.28), (2.31) and (2.32) give now (2.27) via the agm inequality.□

For the k^4 term $\Pi_n^{(2)}$ in (2.25) we have

LEMMA 2.8. *Let* $m\ge 1$, $m+2\le l\le J-1$, *and* $\hat{U}^i$, $m+1\le i\le l+1$, U^i, $m-1\le i\le l$, *exist in* $S_h\cap V$. *Assume that* (1.9, $j=0,1$) *and* (1.4) *hold and that there exists* $a>0$ *such that* $kh^{-1}\le a$. *Then*

$$(2.33)\quad \Big|\sum_{n=m+1}^{l-1}(\Pi_n^{(2)},E^n)\Big|\le ck\sum_{n=m+1}^{l-1}\{\|\hat{e}^{n+2}-\hat{e}^n\|^2\ (1+|\hat{e}^n|_\infty^2)$$
$$+k^2(\|\hat{e}^{n+2}\|^2+\|\hat{e}^n\|^2+\sum_{j=-2}\|e^{n+j}\|^2)+k^2h^{2r}(1+|e^{n-1}|_\infty^2+|e^{n-2}|_\infty^2)$$
$$+(1+|e^{n-1}|_\infty^2)\|E^{n+1}-E^{n-1}\|^2+(1+|e^{n-2}|_\infty^2)\|E^n-E^{n-2}\|^2$$
$$+h^{-2}(|\hat{e}^{n+2}|_\infty^2+|\hat{e}^n|_\infty^2+\sum_{j=-2}^{1}|e^{n+j}|_\infty^2)k^2\|L_n^{1/2}E^n\|^2$$
$$+k^2\|L_n^{1/2}E^n\|^2+k^4\|L_nE^n\|^2\}.$$

<u>Proof</u>. For $1\le i\le J-1$, $g^{i\pm 1}\in Y$, consider the identity

$$(2.34)\quad (L_{i+1}{}^2-L_{i+1}{}^2(g^{i+1}))W^{i+1}-(L_{i-1}{}^2-L_{i-1}{}^2(g^{i-1}))W^{i-1}$$
$$=L_{i+1}(g^{i+1})[L_{i+1}-L_{i+1}(g^{i+1})-L_{i-1}+L_{i-1}(g^{i-1})]W^{i-1}$$
$$+(L_{i+1}(g^{i+1})-L_{i-1}(g^{i-1}))(L_{i-1}-L_{i-1}(g^{i-1}))W^{i-1}$$
$$+L_{i+1}(g^{i+1})(L_{i+1}-L_{i+1}(g^{i+1}))(W^{i+1}-W^{i-1})$$
$$+(L_{i+1}-L_{i+1}(g^{i+1}))(L_{i+1}W^{i+1}-L_{i-1}W^{i-1})$$
$$+(L_{i+1}-L_{i+1}(g^{i+1})-L_{i-1}+L_{i-1}(g^{i-1}))L_{i-1}W^{i-1}.$$

For $\varphi\in S_h$, using (v.e), (1.9), inverse assumptions and (2.1), we have

$$(2.35)\quad |(L_{i+1}(g^{i+1})[L_{i+1}-L_{i+1}(g^{i+1})-L_{i-1}+L_{i-1}(g^{i-1})]W^{i-1},\varphi)|$$
$$\le ch^{-1}[\|u^{i+1}-g^{i+1}-u^{i-1}+g^{i-1}\|(1+|u^{i-1}-g^{i-1}|_\infty)+k\|u^{i-1}-g^{i-1}\|]$$
$$[\|L_{i+1}\varphi\|+ch^{-1}|u^{i+1}-g^{i+1}|_\infty\ \|L_{i+1}{}^{1/2}\varphi\|].$$

Similarly, using also (v.d),

$$(2.36)\quad |((L_{i+1}(g^{i+1})-L_{i-1}(g^{i-1}))(L_{i-1}-L_{i-1}(g^{i-1}))W^{i-1},\varphi)|$$
$$=|((L_{i-1}-L_{i-1}(g^{i-1}))W^{i-1},(L_{i+1}(g^{i+1})-L_{i-1}(g^{i-1}))\varphi)|$$
$$\le ch^{-1}\|u^{i-1}-g^{i-1}\|(k\|L_{i+1}\varphi\|+h^{-1}|u^{i+1}-g^{i+1}|_\infty\ \|L_{i+1}{}^{1/2}\varphi\|$$
$$+h^{-1}|u^{i-1}-g^{i-1}|_\infty\ \|L_{i-1}{}^{1/2}\varphi\|).$$

We also conclude in a similar way for $\varphi\in S_h$:

$$(2.37)\quad |(L_{i+1}(g^{i+1})(L_{i+1}-L_{i+1}(g^{i+1}))(W^{i+1}-W^{i-1}),\varphi)|$$
$$=|((L_{i+1}-L_{i+1}(g^{i+1}))(W^{i+1}-W^{i-1}),(L_{i+1}(g^{i+1})-L_{i+1})\varphi+L_{i+1}\varphi)|$$
$$\le ckh^{-1}\ \|u^{i+1}-g^{i+1}\|(\|L_{i+1}\varphi\|+ch^{-1}|u^{i+1}-g^{i+1}|_\infty\ \|L_{i+1}{}^{1/2}\varphi\|).$$

By (v.d), (1.4) we have for $\varphi\in S_h$, $kh^{-1}\le a$,

$$(2.38)\quad |((L_{i+1}-L_{i+1}(g^{i+1}))(L_{i+1}W^{i+1}-L_{i-1}W^{i-1}),\varphi)|$$
$$\le c\|u^{i+1}-g^{i+1}\|\ \|P\int_{t_{i-1}}^{t_{i+1}}(d/d\xi)(L(\xi)u(\xi))d\xi\|_{1,\infty}\ \|L_{i+1}{}^{1/2}\varphi\|$$
$$\le c\|u^{i+1}-g^{i+1}\|\ \|L_{i+1}{}^{1/2}\varphi\|.$$

Finally, by (v.e), (1.4), $kh^{-1}\le a$, for $\varphi\in S_h$,

$$(2.39)\quad |((L_{i+1}-L_{i+1}(g^{i+1})-L_{i-1}+L_{i-1}(g^{i-1}))L_{i-1}W^{i-1},\varphi)|$$
$$\le ck^{-1}[\|u^{i+1}-g^{i+1}-u^{i-1}+g^{i-1}\|(1+|u^{i-1}-g^{i-1}|_\infty)$$
$$+k\|u^{i-1}-g^{i-1}\|]\|L_{i+1}{}^{1/2}\varphi\|.$$

Applying these estimates for $i=n+1$ and $g^i=\hat{U}^i$, $i=n$ and $g^i=U^i$, $i=n-1$ and $g^i=U^i$ with $m+1\le n\le l-1$, yields, by use of (2.26) and the agm inequality, the desired (2.33).□

We remark for later use that if $l=m$, $1\le m\le J-1$, we simply have $\sum_{n=m}^{l}(S_nW^n,\ E^{n+1}-E^{n-1})=(S_mW^m,E^{m+1})-(S_mW^m,E^{m-1})$. Thus if $\hat{U}^{m+1},U^m,U^{m-1}$ exist in $S_h\cap Y$ and U^{m+1} exists in S_h, and if (1.9, j=0), (1.4) hold and there exists $a>0$ such that $kh^{-1}\le a$, it is easily seen, with estimation techniques similar to the ones used above, that for any $\epsilon_1,\epsilon_2>0$ there exists a constant $c(\epsilon_1,\epsilon_2)>0$ such that

(2.40) $|(S_m W^m, E^{m+1}-E^{m-1})| \le c(\epsilon_1,\epsilon_2)k^2[\|\hat{e}^{m+1}\|^2(1+|\hat{e}^{m+1}|_\infty^2)$

$+\sum_{j=m-1}^{m} \|e^j\|^2(1+|e^j|_\infty^2)]+\epsilon_1 k^2(\|L_{m+1}{}^{1/2}E^{m+1}\|^2+\|L_{m+1}{}^{1/2}E^{m-1}\|^2)$

$+\epsilon_2 k^4(\|L_{m+1}E^{m+1}\|^2+\|L_{m+1}E^{m-1}\|^2)$.

We finally attack the Λ term in the right-hand side of (2.18). We recall that Λ is defined by (2.16) and write

(2.41) $\Lambda(\hat{U}^{n+1},U^n,U^{n-1})=\Lambda_n{}^{(1)}+\Lambda_n{}^{(2)}+(q_1-1/12)(\Lambda_n{}^{(3)}+\Lambda_n{}^{(4)}+\Lambda_n{}^{(5)})$,

$1\le m\le n\le l\le J-1$,

where, for $m\le n\le l$,

(2.42) $\Lambda_n{}^{(1)}=k^2[(q_1 f^{n+1}(\hat{U}^{n+1})-2p_1 f^n(U^n)+q_1 f^{n-1}(U^{n-1}))$

$-(q_1 f^{n+1}-2p_1 f^n+q_1 f^{n-1})]$,

(2.43) $\Lambda_n{}^{(2)}=k^4[(q_2 L_{n+1}(\hat{U}^{n+1})f^{n+1}(\hat{U}^{n+1})-2p_2 L_n(U^n)f^n(U^n)$

$+q_2 L_{n-1}(U^{n-1})f^{n-1}(U^{n-1}))-(q_2 L_{n+1}f^{n+1}-2p_2 L_n f^n+q_2 L_{n-1}f^{n-1})]$,

(2.44) $\Lambda_n{}^{(3)}=-k^4(\delta^2 f^n(\hat{U}^{n+1},U^n,U^{n-1})-f^{(2)n})$,

(2.45) $\Lambda_n{}^{(4)}=k^4(\delta^2 L_n(\hat{U}^{n+1},U^n,U^{n-1})U^n-PL^{(2)}(t_n)u^n)$,

(2.46) $\Lambda_n{}^{(5)}=k^4\{2\delta L_n(\hat{U}^{n+1},U^{n-1})[k^{-1}(U^n-U^{n-1})$

$+(k/2)(-L_n(U^n)U^n+f^n(U^n))]-2PL^{(1)}(t_n)u^{(1)n}\}$.

We shall estimate the terms in (2.18) corresponding to terms $\Lambda_n{}^{(i)}$, $1\le i\le 5$, in the series of lemmata that follow.

LEMMA 2.9. <u>Let</u> $1\le m\le l\le J-1$, <u>and suppose that</u> $\hat{U}^{n+1}$, $m\le n\le l$ <u>and</u> U^n, $m-1\le n\le l$, <u>exist in</u> $S_h\cap Y$ <u>and that</u> U^{l+1} <u>exists in</u> S_h. <u>Then</u>

(2.47) $|\sum_{n=m}^{l}{}' (\Lambda_n{}^{(1)},E^{n+1}-E^{n-1})| \le ck \sum_{n=m}^{l}{}' (k^2(\|\hat{e}^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2)$

$+\|E^{n+1}-E^{n-1}\|^2)$.

<u>Proof</u>. Immediate using the fact that f is Lipschitz and the definition of $\Lambda_n{}^{(1)}$.□

LEMMA 2.10. <u>Let</u> $1\le m\le J-1$, <u>suppose that</u> $\hat{U}^{n+1}$, $m\le n\le l$ <u>and</u> U^n, $m-1\le n\le l$ <u>exist in</u> $S_h\cap Y$, <u>that</u> U^{l+1} <u>exists in</u> S_h, <u>that there exists</u> $a>0$ <u>such that</u> $kh^{-1}\le a$ <u>and that</u> (1.4) <u>holds</u>. <u>Then</u>

(2.48) $|\sum_{n=m}^{l}{}' (\Lambda_n{}^{(2)},E^{n+1}-E^{n-1})| \le ck \sum_{n=m}^{l}{}' (k^2(\|\hat{e}^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2)$

$+k^4h^{-2}(|\hat{e}^{n+1}|_\infty^2+|e^n|_\infty^2+|e^{n-1}|_\infty^2)\ \|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|^2$

$+k^2\|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|^2+k^4\|L_n(E^{n+1}-E^{n-1})\|^2)$.

<u>Proof</u>. For $1\le i\le J-1$, $g^i\in Y$, consider the identity

(2.49) $L_i(g^i)f^i(g^i)-L_i f^i=(L_i(g^i)-L_i)f^i(g^i)+L_i(f^i(g^i)-f^i)$.

We then have for $\varphi\in S_h$, using (v.c), (iv.a), (v.d), the Lipschitz condition on f and (1.4), that

(2.50) $|((L_i g^i)-L_i)f^i(g^i),\varphi)|=|((L_i(g^i)-L_i)(f^i(g^i)-f^i),\varphi)$

$+((L_i(g^i)-L_i)f^i,\varphi)| \le c\|g^i-u^i\|\ \|L_i{}^{1/2}\varphi\|\ (h^{-1}|g^i-u^i|_\infty+k^{-1})$.

Similarly,

$$(2.51)\quad |(L_l(f^l(g^l)-f^l),\varphi)|=|(f^l(g^l)-f^l,L_l\varphi)|\le c\|g^l-u^l\|\,\|L_l\varphi\|.$$

Use of (2.49)-(2.51) for $i=n+1$ and $g^l=\hat{U}^{n+1}$, $i=n$ and $g^l=U^n$, $i=n-1$ and $g^l=U^{n-1}$ and of the agm inequality now yields (2.48).□

LEMMA 2.11. <u>Let</u> $1\le m\le l\le J$ <u>and suppose that</u> $\hat{U}^{n+1}$, $m\le n\le l$ <u>and</u> U^n, $m-1\le n\le l$ <u>exist in</u> $S_h\cap Y$ <u>and that</u> U^{l+1} <u>exists in</u> S_h. <u>Then</u>

$$(2.52)\quad |(\textstyle\sum_{n=m}^{l}(\Lambda_n^{(3)},E^{n+1}-E^{n-1})|\le ck^{10}(l-m+1)k+ck\sum_{n=m}^{l}\{k^2(\|\hat{e}^{n+1}\|^2$$
$$+\|e^n\|^2+\|e^{n-1}\|^2)+\|E^{n+1}-E^{n-1}\|^2\}.$$

<u>Proof</u>. Write $\Lambda_n^{(3)}=-k^4(k^{-2}[(f^{n+1}(\hat{U}^{n+1})-2f^n(U^n)+f^{n-1}(U^{n-1}))$ $-(f^{n+1}-2f^n+f^{n-1})]+k^{-2}(f^{n+1}-2f^n+f^{n-1})-f^{(2)n})$, from which (2.52) follows, using Taylor's theorem and the smoothness of f.□

LEMMA 2.12. <u>Let</u> $m\ge 1$ <u>and</u> $m+2\le l\le J-1$. <u>Suppose that</u> $\hat{U}^{n+1}$, $m\le n\le l$ <u>and</u> U^n, $m-1\le n\le l$ <u>exist</u> in $S_h\cap Y$, <u>that</u> U^{l+1} <u>exists in</u> S_h <u>and that</u> (1.9) <u>holds</u>. <u>Then</u>, <u>for any</u> $\varepsilon_3>0$, <u>there exists a constant</u> $c(\varepsilon_3)>0$ <u>such that</u>

$$(2.53)\quad |\textstyle\sum_{n=m}^{l}(\Lambda_n^{(4)},E^{n+1}-E^{n-1})|\le\eta_m^{(3)}+ck^2(k^4+h^r)^2(l-m+1)k$$
$$+\varepsilon_3k^2(\|L_{l+1}{}^{1/2}(E^{l+1}+E^l)\|^2+\|L_{l+1}{}^{1/2}(E^{l+1}-E^l)\|^2)$$
$$+c(\varepsilon_3)k^2(\textstyle\sum_{j=l}^{l+1}\|\hat{e}^j\|^2+\sum_{j=l-2}^{l}\|e^j\|^2)$$
$$+ck\textstyle\sum_{n=m}^{l}\{\|E^{n+1}-E^{n-1}\|^2+h^{-2}(|\hat{e}^{n+1}|_\infty^2+|e^n|_\infty^2+|e^{n-1}|_\infty^2)\|E^{n+1}-E^{n-1}\|^2$$
$$+k^2(\|L_n{}^{1/2}E^n\|^2+\|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|^2)+k^4\|L_n(E^{n+1}-E^{n-1})\|^2\}$$
$$+ck\textstyle\sum_{n=m+1}^{l-1}\{k^2(\|\hat{e}^{n+2}\|^2+\|\hat{e}^n\|^2+\sum_{j=n-2}^{n+1}\|e^j\|^2)$$
$$+\|\hat{e}^{n+2}-\hat{e}^n\|^2(1+|\hat{e}^n|_\infty^2)+(1+|e^{n-1}|_\infty^2)\|E^{n+1}-E^{n-1}\|^2$$
$$+(1+|e^{n-2}|_\infty^2)\,\|E^n-E^{n-2}\|^2+k^2h^{2r}(1+|e^{n-1}|_\infty^2+|e^{n-2}|_\infty^2)\},$$

<u>where</u>

$$(2.54)\quad \eta_m^{(3)}=ck^2(\textstyle\sum_{j=m+1}^{m+2}\|\hat{e}^j\|^2+\sum_{j=m-1}^{m+1}\|e^j\|^2)+k^2(\|L_m{}^{1/2}(E^m+E^{m-1})\|^2$$
$$+\|L_m{}^{1/2}(E^m-E^{m-1})\|^2).$$

<u>Proof</u>. Using the notation $\delta^2v_n=k^{-2}(v_{n+1}-2v_n+v_{n-1})$, write

$$(2.55)\quad \Lambda_n^{(4)}=k^4[(\delta^2L_nW^n-L^{(2)}{}_nW^n)+(L^{(2)}{}_nW^n-PL^{(2)}(t_n)u^n)$$
$$+(\delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})U^n-\delta^2L_nW^n)].$$

Using an integral representation of $\delta^2L_n-L^{(2)}{}_n$ we see, by (1.7), that

$$(2.56)\quad |\textstyle\sum_{n=m}^{l}k^4(\delta^2L_nW^n-L^{(2)}{}_nW^n,E^{n+1}-E^{n-1})|$$
$$\le c\textstyle\sum_{n=m}^{l}(k^6\|E^{n+1}-E^{n-1}\|\sup_{\xi\in[t_{n-1},t_{n+1}]}\|L_h{}^{(4)}(\xi)W^n\|$$
$$\le ck\textstyle\sum_{n=m}^{l}(k^{10}+\|E^{n+1}-E^{n-1}\|^2).$$

Using estimates entirely analogous to the ones that led to (2.15) of Lemma 2.2 of [3], since $L^{(2)}{}_nW^n-PL^{(2)}(t_n)u^n=(L^{(2)}{}_nT_n-PL^{(2)}(t_n)T(t_n))L(t_n)u^n$, we conclude, without requiring $L(t^n)u^n\in D_L$, that

$$(2.57)\quad |\textstyle\sum_{n=m}^{l}k^4(L^{(2)}{}_nW^n-PL^{(2)}(t_n)u^n,E^{n+1}-E^{n-1})|$$
$$\le ck\textstyle\sum_{n=m}^{l}(k^2h^{2r}+k^4\|L_n(E^{n+1}-E^{n-1})\|^2).$$

We now write the last term of the right-hand side of (2.55) as

$$(2.58)\quad \delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})U^n-\delta^2L_nW^n=(\delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})-\delta^2L_n)E^n$$
$$+(\delta^2L_n)E^n+(\delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})-\delta^2L_n)W^n.$$

Observe that by (v.c) the agm inequality and inverse assumptions,

$$(2.59)\quad |\sum_{n=m}^{l} k^4((\delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})-\delta^2L_n)E^n,E^{n+1}-E^{n-1})|$$
$$\le ck\sum_{n=m}^{l}(h^{-2}(|\hat{e}^{n+1}|_\infty^2+|e^n|_\infty^2+|e^{n-1}|_\infty^2)\|E^{n+1}-E^{n-1}\|^2+k^2\|L_n^{1/2}E^n\|^2).$$

Using now the estimate $|(\delta^2L_n\varphi,\psi)|\le c\|L_n^{1/2}\varphi\|\,\|L_n^{1/2}\psi\|$, $\varphi,\psi\in S_h$, which is easily established by an integral representation of δ^2L_n, it is seen that

$$(2.60)\quad |\sum_{n=m}^{l} k^4((\delta^2L_n)E^n,E^{n+1}-E^{n-1})|\le ck^2\sum_{n=m}^{l}(k^2(\|L_n^{1/2}E^n\|^2$$
$$+\|L_n^{1/2}(E^{n+1}-E^{n-1})\|^2)).$$

For the last term in the right-hand side of (2.58), writing $\Xi^n=(\delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})-\delta^2L_n)W^n$, $m\le n\le l$, and using summation by parts, we can write, since $l\ge m+2$

$$(2.61)\quad \sum_{n=m}^{l} k^4((\delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})-\delta^2L_n)W^n,E^{n+1}-E^{n-1})$$
$$=k^4\sum_{n=m}^{l}(\Xi^n,E^{n+1}-E^{n-1})=k^4[(\Xi^l,E^{l+1})+(\Xi^{l-1},E^l)-(\Xi^m,E^{m-1})$$
$$-(\Xi^{m+1},E^m)-\sum_{n=m+1}^{l-1}(\Xi^{n+1}-\Xi^{n-1},E^n)].$$

It is not hard to see, using (v.d) and (1.9), that for any $\epsilon_3>0$ there exists a constant $c(\epsilon_3)>0$ such that

$$(2.62)\quad |k^4[(\Xi^l,E^{l+1})+(\Xi^{l-1},E^l)]|\le c(\epsilon_3)k^2(\sum_{j=l+1}^{l+2}\|\hat{e}^j\|^2+\sum_{j=l-2}^{l}\|e^j\|^2)$$
$$+\epsilon_3k^2(\|L_{l+1}^{1/2}(E^{l+1}+E^l)\|^2+\|L_{l+1}^{1/2}(E^{l+1}-E^l)\|^2)$$

and that, with $\eta_m^{(3)}$ defined by (2.54),

$$(2.63)\quad |k^4[(\Xi^m,E^{m-1})+(\Xi^{m+1},E^m)]|\le\eta_m^{(3)}.$$

To simplify notation for treating the last term in the right-hand side of (2.61), define the operator $K_n=\delta^2L_n(\hat{U}^{n+1},U^n,U^{n-1})-\delta^2L_n$, so that $\Xi^n=K_nW^n$. Then, since

$$(2.64)\quad k^4\sum_{n=m+1}^{l-1}(\Xi^{n+1}-\Xi^{n-1},E^n)=k^4\sum_{n=m+1}^{l-1}((K_{n+1}(W^{n+1}-W^{n-1}),E^n)$$
$$+((K_{n+1}-K_{n-1})W^{n-1},E^n)),$$

we have by (v.d), (1.9) that

$$(2.65)\quad |k^4\sum_{n=m+1}^{l-1}(K_{n+1}(W^{n+1}-W^{n-1}),E^n)|\le ck\sum_{n=m+1}^{l-1}(k^2(\|\hat{e}^{n+2}\|^2$$
$$+\|e^{n+1}\|^2+\|e^n\|^2+\|L_n^{1/2}E^n\|^2)).$$

Finally, by (2.30), (1.9), we obtain, using estimates similar to those in the proof of Lemma 2.8 that

$$(2.66)\quad |k^4 \sum_{n=m+1}^{l-1} ((K_{n+1}-K_{n-1})W^{n-1},E^n)| \le |\sum_{n=m+1}^{l-1} k^2(([L_{n+2}(\hat{U}^{n+2})$$
$$-L_{n+2}-L_n(\hat{U}^n)+L_n]-2[L_{n+1}(U^{n+1})-L_{n+1}-L_{n-1}(U^{n-1})+L_{n-1}]$$
$$+[L_n(U^n)-L_n-L_{n-2}(U^{n-2})+L_{n-2}])W^{n-1},E^n)| \le ck\sum_{n=m+1}^{l-1}(k^2(\|\hat{e}^n\|^2$$
$$+\|e^{n-1}\|^2+\|e^{n-2}\|^2+\|L_n^{1/2}E^n\|^2)+\|\hat{e}^{n+2}-\hat{e}^n\|^2(1+|\hat{e}^n|_\infty^2)$$
$$+k^2h^r(1+|e^{n-1}|_\infty^2+|e^{n-2}|_\infty^2)+(1+|e^{n-1}|_\infty^2)\|E^{n+1}-E^{n-1}\|^2$$
$$+(1+|e^{n-2}|_\infty^2)\|E^n-E^{n-2}\|^2).$$

Collecting terms from (2.55)-(2.66), we obtain (2.53).□

This lemma required that $l\ge m+2$. For later use, let us also remark that in the case $l=m$, $1\le m\le J-1$, assuming that $\hat{U}^{m+1}, U^{m-1}, U^m$ exist in $S_h\cap Y$, that U^{m+1} exists in S_h, and that (1.9) holds, one may similarly prove that given $\epsilon_4,\epsilon_5>0$ there exists a constant $c(\epsilon_4,\epsilon_5)>0$ such that

$$(2.67)\quad |(A_m^{(4)},E^{m+1}-E^{m-1})| \le ck^3(k^4+h^r)^2+ck[\|E^{m+1}-E^{m-1}\|^2$$
$$+k^2(\|L_m^{1/2}E^m\|^2+\|L_m^{1/2}(E^{m+1}-E^{m-1})\|^2)+k^4\|L_m(E^{m+1}-E^{m-1})\|^2]$$
$$+ckh^{-2}(|\hat{e}^{m+1}|_\infty^2+|e^m|_\infty^2+|e^{m-1}|_\infty^2)\|E^{m+1}-E^{m-1}\|^2$$
$$+c(\epsilon_4,\epsilon_5)k^2(\|\hat{e}^{m+1}\|^2+\|e^m\|^2+\|e^{m-1}\|^2)+\epsilon_4k^2\|L_m^{1/2}E^{m+1}\|^2$$
$$+\epsilon_5k^2\|L_m^{1/2}E^{m-1}\|^2 .$$

In the following final lemma we estimate the $A_n^{(5)}$ term.

LEMMA 2.13. <u>Let</u> $1\le m\le J-1$ <u>and suppose that</u> $\hat{U}^{n+1}$, $m\le n\le l$ <u>and</u> U^n, $m-1\le n\le l$ <u>exist in</u> $S_h\cap Y$ <u>and that</u> U^{l+1} <u>exists in</u> S_h. <u>Assume that</u> (1.9) <u>and</u> (1.4) <u>hold and that there exists</u> $a>0$ <u>such that</u> $kh^{-1}\le a$. <u>Then</u>

$$(2.68)\quad |(\sum_{n=m}^{l}(A_n^{(5)},E^{n+1}-E^{n-1})| \le ck^2(k^4+h^r)^2(l-m+1)k$$
$$+ck\sum_{n=m}^{l}(\|E^{n+1}-E^{n-1}\|^2$$
$$+k^2(\|L_{n+1}^{1/2}(E^{n+1}-E^{n-1})\|^2+\|L_{n+1}^{1/2}(E^n-E^{n-1})\|^2)$$
$$+k^4(\|L_nE^n\|^2+\|L_{n+1}(E^{n+1}-E^{n-1})\|^2)+k^2(\|\hat{e}^{n+1}\|^2+\|e^n\|^2+\|e^{n-1}\|^2)$$
$$+h^{-2}(|\hat{e}^{n+1}|_\infty^2+|e^{n-1}|_\infty^2)(\|E^n-E^{n-1}\|^2+\|E^{n+1}-E^{n-1}\|^2)$$
$$+k^2h^{-2}|e^n|_\infty^2\|L_n^{1/2}E^n\|^2).$$

<u>Proof</u>. We write

$$(2.69)\quad A_n^{(5)}=2(I_n^{(1)}+I_n^{(2)}),$$

where for $m\le n\le l$,

$$(2.70)\quad I_n^{(1)}=k^4(L^{(1)}{}_n[k^{-1}(W^n-W^{n-1})+(k/2)(-L_nW^n+f^n)]-PL^{(1)}(t_n)u^{(1)n}),$$

$$(2.71)\quad I_n^{(2)}=k^4(\delta L_n(\hat{U}^{n+1},U^{n-1})[k^{-1}(U^n-U^{n-1})+(k/2)(-L_n(U^n)U^n+f^n(U^n))]$$
$$-L^{(1)}{}_n[k^{-1}(W^n-W^{n-1})+(k/2)(-L_nW^n+f^n)]).$$

Noting that $-L_nW^n+f^n=Pu^{(2)n}$, we have

$$(2.72)\quad I_n^{(1)}=k^4(L^{(1)}{}_n[k^{-1}(W^n-W^{n-1})-k^{-1}P(u^n-u^{n-1})])$$
$$+k^4(L^{(1)}{}_nP[(k/2)u^{(2)n}+k^{-1}(u^n-u^{n-1})-u^{(1)n}])$$
$$+k^4(L^{(1)}{}_nPu^{(1)n}-PL^{(1)}(t_n)u^{(1)n})\equiv J_n^{(1)}+J_n^{(2)}+J_n^{(3)}.$$

For $J_n^{(1)}$ we have, using (1.6) and (1.7), that

$$(2.73)\quad |\sum_{n=m}^{l}(J_n^{(1)},E^{n+1}-E^{n-1})| \le c\sum_{n=m}^{l}k^4h^r\|L_n(E^{n+1}-E^{n-1})\|.$$

Denoting $z^n=(k/2)u^{(2)n}+k^{-1}(u^n-u^{n-1})-u^{(1)n}$ and observing that $z^n\in D_L$ and, because u is smooth, $\|z^n\|_j\leq c_jk^2$, $j=0,1,2,\ldots$, we obtain by (1.2), with $P_1{}^n=P_1(t^n)$,

$$(2.74)\quad |\textstyle\sum'_{n=m}(J_n{}^{(2)},E^{n+1}-E^{n-1})|=|\sum'_{n=m}k^4(L^{(1)}{}_n[P_1{}^nz^n+(P-P_1{}^n)z^n],$$
$$E^{n+1}-E^{n-1})|\leq\textstyle\sum'_{n=m}\{k^4(\|L^{(1)}{}_nT_n\|\;\|L(t_n)z^n\|\;\|E^{n+1}-E^{n-1}\|$$
$$+\|T_nL^{(1)}{}_nP\|\;\|(P-P_1{}^n)z^n\|\;\|L_n(E^{n+1}-E^{n-1})\|\}$$
$$\leq c\textstyle\sum'_{n=m}\{k^6\|E^{n+1}-E^{n-1}\|+k^6h^r\;\|L_n(E^{n+1}-E^{n-1})\|\}.$$

By the analog to Lemma 2.2 of [3] and (1.2) we see that

$$(2.75)\quad |\textstyle\sum'_{n=m}(J_n{}^{(3)},E^{n+1}-E^{n-1})|=|\sum'_{n=m}k^4([L^{(1)}{}_n(P-P_1{}^n)+(L^{(1)}{}_nT_n-$$
$$-PL^{(1)}(t_n)T(t_n))L(t^n)]u^{(1)n},E^{n+1}-E^{n-1})|$$
$$\leq c\textstyle\sum'_{n=m}k^4h^r\|L_n(E^{n+1}-E^{n-1})\|.$$

We now proceed to the term $I_n{}^{(2)}$ which we write as

$$(2.76)\quad I_n{}^{(2)}=k^4(\delta L_n(\hat{U}^{n+1},U^{n-1})(k^{-1}(U^n-U^{n-1}))-L^{(1)}{}_n(k^{-1}(W^n-W^{n-1})))$$
$$+(k^5/2)(\delta L_n(\hat{U}^{n+1},U^{n-1})(-L_n(U^n)U^n+f^n(U^n))-L^{(1)}{}_n(-L_nW^n+f^n))$$
$$\equiv M_n{}^{(1)}+M_n{}^{(2)}.$$

Denoting $\delta v^n=(2k)^{-1}(v^{n+1}-v^{n-1})$ for vector- or operator-valued v, write $M_n{}^{(1)}$ as

$$(2.77)\quad M_n{}^{(1)}=k^4[(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)(k^{-1}(W^n-W^{n-1}))]$$
$$+k^4[(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)(k^{-1}(E^n-E^{n-1}))]$$
$$+k^4[(\delta L_n)k^{-1}(E^n-E^{n-1})]+k^4[(\delta L_n-L^{(1)}{}_n)k^{-1}(W^n-W^{n-1})]$$
$$\equiv M_n{}^{(1.1)}+M_n{}^{(1.2)}+M_n{}^{(1.3)}+M_n{}^{(1.4)}.$$

By (v.d), (1.9, j=1) we obtain

$$(2.78)\quad |\textstyle\sum'_{n=m}(M_n{}^{(1.1)},E^{n+1}-E^{n-1})|$$
$$\leq c\textstyle\sum'_{n=m}\{k^3(\|\hat{e}^{n+1}\|+\|e^{n-1}\|)\|L_{n+1}{}^{1/2}(E^{n+1}-E^{n-1})\|\}.$$

Similarly by (v.c) and inverse assumptions,

$$(2.79)\quad |\textstyle\sum'_{n=m}(M_n{}^{(1.2)},E^{n+1}-E^{n-1})|\leq c\sum'_{n=m}\{k^2h^{-1}(|\hat{e}^{n+1}|_\infty$$
$$+|e^{n-1}|_\infty)\|E^n-E^{n-1}\|\|L_{n+1}{}^{1/2}(E^{n+1}-E^{n-1})\|\}.$$

Using $|(\delta L_n\varphi,\psi)|\leq c\|L_h{}^{1/2}(s)\varphi\|\;\|L_h{}^{1/2}(t)\psi\|$ for $\varphi,\psi\in S_h$, which is easily established by an integral representation, we have

$$(2.80)\quad |\textstyle\sum'_{n=m}(M_n{}^{(1.3)},E^{n+1}-E^{n-1})|$$
$$\leq c\textstyle\sum'_{n=m}k^3\|L_{n+1}{}^{1/2}(E^n-E^{n-1})\|\;\|L_{n+1}{}^{1/2}(E^{n+1}-E^{n-1})\|.$$

We also have by Taylor's theorem and (1.7) that

$$(2.81)\quad |\textstyle\sum'_{n=m}(M_n{}^{(1.4)},E^{n+1}-E^{n-1})|\leq c\sum'_{n=m}k^6\|E^{n+1}-E^{n-1}\|.$$

We now write

$$(2.82)\quad \Pi_n^{(2)}=(k^5/2)[(\delta L_n-L^{(1)}{}_n)(-L_nW^n+f^n)]$$
$$+(k^5/2)[\delta L_n-\delta L_n(\hat{U}^{n+1},U^{n-1}))L_nW^n]$$
$$-(k^5/2)[(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)(L_n(U^n)U^n-L_nW^n)]$$
$$-(k^5/2)[(\delta L_n)(L_n(U^n)U^n-L_nW^n)]$$
$$+(k^5/2)[\delta L_n(\hat{U}^{n+1},U^{n-1})f^n(U^n)-(\delta L_n)f^n]\equiv\sum_{j=1}^{5}\Pi_n^{(2.j)}.$$

By (2.1) we obtain

$$(2.83)\quad |\sum'_{n=m}(\Pi_n^{(2.1)},E^{n+1}-E^{n-1})|$$
$$=(k^5/2)\ |\sum'_{n=m}((\delta L_n-L^{(1)}{}_n)Pu^{(2)n},E^{n+1}-E^{n-1})|$$
$$\leq c\sum'_{n=m}k^7\sup_{\xi\in(t_{n-1},t_{n+1})}\|L_h^{(3)}(\xi)T_n\|\ \|L_nPu^{(2)n}\|\|E^{n+1}-E^{n-1}\|$$
$$\leq c\sum'_{n=m}k^7\|E^{n+1}-E^{n-1}\|,$$

where we have used the fact that if $v\in D_L$, then by (iv.a), (1.3), (1.5), $\|L_nPv\|\leq\|L_n(P-P_I{}^n)v\|+\|L(t_n)v\|\leq c\|v\|_2$. By (v.d), $kh^{-1}\leq a$ and (1.4), we now see that

$$(2.84)\quad |\sum'_{n=m}(\Pi_n^{(2.2)},E^{n+1}-E^{n-1})|$$
$$=|c\ \sum'_{n=m}k^4((L_{n+1}(\hat{U}^{n+1})-L_{n+1}-L_{n-1}(U^{n-1})$$
$$+L_{n-1})PL(t_n)u^n,E^{n+1}-E^{n-1})|$$
$$\leq c\ \sum'_{n=m}k^3(\|\hat{e}^{n+1}\|+\|e^{n-1}\|)\|L_{n+1}{}^{1/2}(E^{n+1}-E^{n-1})\|.$$

By (v.c), inverse assumptions, $kh^{-1}\leq a$, (1.9) and (2.1), we also obtain

$$(2.85)\quad |\sum'_{n=m}(\Pi_n^{(2.3)},E^{n+1}-E^{n-1})|=|c\sum'_{n=m}k^5([(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)$$
$$(L_n(U^n)-L_n)W^n]+[(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)(L_n(U^n)-L_n)E^n]$$
$$+[(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)L_nE^n],E^{n+1}-E^{n-1})\ |$$
$$\leq c\sum'_{n=m}\{k^2h^{-1}(|\hat{e}^{n+1}|_\infty+|e^{n-1}|_\infty)\|E^{n+1}-E^{n-1}\|[\|e^n\|$$
$$+kh^{-1}|e^n|_\infty\|L_n{}^{1/2}E^n\|+k\|L_nE^n\|]\}.$$

Using (2.1), (1.9), $kh^{-1}\leq a$ and inverse assumptions, we have

$$(2.86)\quad |\sum'_{n=m}(\Pi_n^{(2.4)},E^{n+1}-E^{n-1})|=|c\sum'_{n=m}k^5([\delta L_n(L_n(U^n)-L_n)W^n]$$
$$+[\delta L_n(L_n(U^n)-L_n)E^n]+[(\delta L_n)L_nE^n],\ E^{n+1}-E^{n-1})|$$
$$\leq c\sum'_{n=m}\{k^3\|e^n\|\ \|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|$$
$$+k^4h^{-1}|e^n|_\infty\ \|L_n{}^{1/2}E^n\|\ \|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|$$
$$+k^4\|L_nE^n\|\ \|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|\}.$$

Finally, by (v.d), (1.4), $kh^{-1}\leq a$ we see that

$$(2.87)\quad |\sum'_{n=m}(\Pi_n^{(2.5)},E^{n+1}-E^{n-1})|$$
$$=|c\sum'_{n=m}k^5([(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)f^n]$$
$$+[(\delta L_n(\hat{U}^{n+1},U^{n-1})-\delta L_n)(f^n(U^n)-f^n)]+[(\delta L_n)(f^n(U^n)-f^n)],E^{n+1}-E^{n-1})|$$
$$\leq\ c\sum'_{n=m}\{k^3(\|\hat{e}^{n+1}\|+\|e^{n-1}\|)\|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|$$
$$+k^3h^{-1}(|\hat{e}^{n+1}|_\infty+|e^{n-1}|_\infty)\|e^n\|\ \|E^{n+1}-E^{n-1}\|+k^4\|e^n\|\ \|L_n{}^{1/2}(E^{n+1}-E^{n-1})\|\}.$$

Collecting all terms, we establish (2.68) from (2.69)-(2.87) and the agm inequality.□

S3. STARTING AND CONVERGENCE OF THE SCHEME

In the sequel we let $Q_n=\tilde{q}(kL_n)$, where L_n has been defined after (3.2), set $e^j=u^j-U^j$, $\hat{e}_1{}^j=u_j-\hat{U}_1{}^j$ and define A_1 by (3.7).

LEMMA 3.1. <u>Suppose that</u> $U^0,\hat{U}_1{}^j$, $1\le j\le 3$, <u>belong to</u> $S_h\cap V$. <u>Then there exists a constant</u> $c>0$ <u>such that for each</u> $\phi\in S_h{}^2$

$$(3.10)\quad |||(A_1-Q_1)\phi|||_1\le c[kh^{-1}(|e^0|_\infty+\Sigma_{j=1}^{3}|\hat{e}_1{}^j|_\infty)+k]\ |||Q_1\phi|||_1 .$$

<u>Proof</u>. Letting $\phi=(\phi_1,\phi_2)^T$ we have, by the definition of A_1,Q_1 and the $|||.|||_1$ norm, that

$$(3.11)\quad |||(A_1-Q_1)\phi|||_1\le k^2||(L_1-L_1(\hat{U}_1{}^1))\phi_1||/12+k||T_1{}^{1/2}(L_1-L_1(\hat{U}_1{}^1))\phi_1||/2$$
$$+(k^2/12)(||T_1{}^{1/2}(L_1-L_1(\hat{U}_1{}^1))\phi_2||+||T_1{}^{1/2}[(-L_3(\hat{U}_1{}^3)$$
$$+6L_2(\hat{U}_1{}^2)-3L_1(\hat{U}_1{}^1)-2L_0(U_1{}^0))/6k]\phi_1||).$$

Now, by (4.8) of [2] and by (2.1) we have

$$(3.12)\quad ||T_1{}^{1/2}(L_i-L_1(\hat{U}_1{}^1))\phi_i||\le c|\hat{e}_1{}^1|_\infty\ ||L_1{}^{1/2}\phi_i||,\quad i=1,2,$$

$$(3.13)\quad ||(L_1-L_1(\hat{U}_1{}^1))\phi_1||\le ch^{-1}|\hat{e}_1{}^1|_\infty\ ||L_1{}^{1/2}\phi_1||.$$

To estimate the last term in the right-hand side of (3.11), we add and subtract the difference quotient $(6k)^{-1}[-(L_3-L_2)+3(L_2-L_1)+2(L_2-L_0)]\phi_1$ and obtain, using (4.8) of [2],

$$(3.14)\quad k^2||T_1{}^{1/2}(6k)^{-1}[-L_3(\hat{U}_1{}^3)+6L_2(\hat{U}_1{}^2)-3L_1(\hat{U}_1{}^1)-2L_0(U_1{}^0)]\phi_1||$$
$$\le ck(|e^0|_\infty+\Sigma_{j=1}^{3}|\hat{e}_1{}^j|_\infty)||L_1{}^{1/2}\phi_1||+ck^2||L_1{}^{1/2}\phi_1||.$$

In (3.14), use was made of the estimate $||T_1{}^{1/2}(L_{n+j}-L_n)\phi_1||$ $\le cjk\sup_{\xi\in[t_n,t_{n+j}]}||T_1{}^{1/2}L_h{}^{(1)}(\xi)T_1{}^{1/2}L_1{}^{1/2}\phi_1||\le cjk\ ||L_1{}^{1/2}\phi_1||$, with the observation that the last inequality - (2.7) of [2]-follows from our assumptions (i)-(iii) of section 1. The desired estimate (3.10) follows now from (3.11)-(3.14), using (iv.a) and (4.3) and (4.4) of [2].□

It is obvious from (3.10) that A_1 will be invertible if, e.g., $k\le ah$ for some $a>0$ and if $|e^0|_\infty$, $|\hat{e}_1{}^j|_\infty$ are sufficiently small, e.g., if they are $\le h$. The next lemma provides us with an appropriate a priori error estimate for U^1, the solution of (3.6). In the sequel we let $W(t)=(u(t),\ u^{(1)}(t))^T$, $W^n=W(t_n)$, $E^n=U^n-W^n$, $n=0,1$.

LEMMA 3.2. <u>Let</u> U^0 <u>be given by</u> (3.3). <u>Suppose that</u> U^1 <u>exists in</u> $S_h{}^2$, <u>that</u> $U^0,\hat{U}_1{}^j$, $1\le j\le 3$, <u>are in</u> $S_h\cap V$, <u>and that</u> (1.7) <u>and</u> (1.9) <u>hold</u>. <u>Then</u>

$$(3.15)\quad |||Q_1E^1|||_1\le ck(k^4+h^r)+ck[||e^0||+\Sigma_{j=1}^{3}\ ||\hat{e}_1{}^j||+kh^{-1}(||e^0||+||\hat{e}_1{}^1||)]$$
$$+c[kh^{-1}(|e^0|_\infty+\Sigma_{j=1}^{3}\ |\hat{e}_1{}^j|_\infty)+k]\ |||Q_1E^1|||_1 .$$

<u>Proof</u>. Taking into account that $E^0=0$ and denoting $\tilde{Q}_n=\tilde{q}(kL_n)+k^2L^{(1)}{}_n/12$, $\tilde{P}_n=\tilde{p}(kL_n)+k^2L^{(1)}{}_n/12$, we obtain the error equation

(3.16) $Q_1E^1=(Q_1-\tilde{Q}_1)E^1+C^1+D^1+(\tilde{Q}_1-A_1)E^1+(\tilde{Q}_1-A_1)W^1+(B_o-\tilde{P}_o)W^o,$

where B_o has been defined by (3.8) and

(3.17) $C^1=-\tilde{Q}_1W^1+\tilde{P}_oW^o+((k^2/12)(f^o-f^1),$
$(k/2)(f^o+f^1)+(k^2/12)(f^{(1)o}-f^{(1)1}))^T$

and

$$(3.18)\ D^1=(k^2/12)\begin{bmatrix}0\\(6k)^{-1}[2f^3(\hat{U}_1{}^3)-9f^2(\hat{U}_1{}^2)+18f^1(\hat{U}_1{}^1)-11f^o]-f^{(1)o}\end{bmatrix}$$

$$+(k/2)\begin{bmatrix}0\\f^1(\hat{U}_1{}^1)-f^1\end{bmatrix}$$

$$-(k^2/12)\begin{bmatrix}f^1(\hat{U}_1{}^1)-f^1\\(6k)^{-1}[-f^3(\hat{U}_1{}^3)+6f^2(\hat{U}_1{}^2)-3f^1(\hat{U}_1{}^1)-2f^o(U^o)]-f^{(1)1}\end{bmatrix}.$$

Now, by [2,(4.13)],

(3.19) $|||(Q_1-\tilde{Q}_1)E^1|||_1\le ck|||Q_1E^1|||_1.$

In addition, [2, Lemma 4.4, (4.14)] provides the consistency estimate

(3.20) $|||C^1|||_1\le ck(k^4+h^r).$

It is not hard to see, since $U^o,\hat{U}_1{}^j$ are in Y, that adding and subtracting the appropriate difference quotients (e.g., $(6k)^{-1}[2f^3-9f^2+18f^1-11f^o]$, etc.) and using the smoothness of f in time, we can also obtain

(3.21) $|||D^1|||_1\le ck(\|e^o\|+\Sigma_{j=1}^3\ \|\hat{e}_1{}^j\|)+ck^5.$

Now, by (3.19) and (3.10),

(3.22) $|||(\tilde{Q}_1-A_1)E^1|||_1\le c[kh^{-1}(|e^o|_\infty+\Sigma_{j=1}^3\ |\hat{e}_1{}^j|_\infty+k]\ |||Q_1E^1|||_1.$

Finally, using our hypotheses, (1.7), (1.9), (2.1) and (4.8), of [2], and adding and subtracting appropriate difference quotients involving L_j, we see that

(3.23) $|||(\tilde{Q}-A_1)W^1+(B_o-\tilde{P}_o)W^o|||_1$
$\le(k/2)(\|T_1{}^{1/2}(-L_1+L_1(\hat{U}_1{}^1))\mu^1\|+\|T_1{}^{1/2}(-L_o(U^o)+L_o)\mu^o\|)$
$+(k^2/12)(\|(-L_1+L_1(\hat{U}_1{}^1))\mu^1\|+\|T_1{}^{1/2}(-L_1+L_1(\hat{U}_1{}^1))\mu^{(1)1}\|$
$+\|(-L_o(U^o)+L_o)\mu^o\|+\|T_1{}^{1/2}(-L_o(U^o)+L_o)\mu^{(1)o}\|$
$+\|T_1{}^{1/2}(-L^{(1)}{}_1+(6k)^{-1}[-L_3(\hat{U}_1{}^3)+6L_2(\hat{U}_1{}^2)-3L_1(\hat{U}_1{}^1)-2L_o(U^o)])\mu^1\|$
$+\|T_1{}^{1/2}(L^{(1)}{}_o-(6k)^{-1}[2L_3(\hat{U}_1{}^3)-9L_2(\hat{U}_1{}^2)+18L_1(\hat{U}_1{}^1)-11L_o(U^o)])\mu^o\|)$
$\le ck[\|\hat{e}_1{}^2\|+\|\hat{e}_1{}^3\|+(1+kh^{-1})(\|\hat{e}_1{}^1\|+\|e^o\|)+k^4].$

(3.16)-(3.23) now yield (3.15).□

Putting together the results of these two lemmata, we obtain

PROPOSITION 3.1. _Let_ $U^0, U^0, \hat{U}_1^j$, $1\le j\le 3$, _be given by_ (3.1), (3.3), (3.4). _Suppose that there exists_ $a>0$ _such that_ $kh^{-1}\le a$ _and let_ k,h _be sufficiently small_. _Then_ U^1, _the solution of_ (3.6), _exists uniquely_. _If_ U^1 _is given by_ (3.5) _and we assume that_ (1.7) _and_ (1.9) _hold_, _it follows that there exists a constant_ $c>0$ _such that_

(3.24) $|||Q_1E^1|||_1\le ck(k^4+h^r)$,

(3.25) $\|U^1-W^1\|+k\|L_1{}^{1/2}(U^1-W^1)\|+k^2\|L_1(U^1-W^1)\|\le ck(k^4+h^r)$.

Proof. By (3.1) and (3.4) we have for $1\le j\le 3$, using (1.3), that

(3.26) $\|\hat{e}_1{}^j\|=\|(W^0-u^0)+[u^0+jku^{(1)0}+(jk)^2u^{(2)0}/2!+(jk)^3u^{(3)0}/3!-u^j]$
$+(P-I)[jku^{(1)0}+(jk)^2u^{(2)0}/2!+(jk)^3u^{(3)0}/3!]\|\le c(k^4+h^r)$.

Note that for $v\in D_L$ and sufficiently smooth, we have, by (iv.b), (ii.b), (1.3) and (1.5), since $r\ge 2$, $1\le N\le 3$, that for $t\in[0,t^*]$:

(3.27) $|(P-I)v|_\infty\le|(P-P_1(t))v|_\infty+|(T_h(t)-T(t))L(t)v|_\infty$
$\le ch^{r-N/2}+ch^r|\ln h|^{\bar{r}}\le ch^{1/2}$.

Since by (1.8) for $r\ge 2$, $1\le N\le 3$, $|U^0-u^0|_\infty=|W^0-u^0|_\infty\le ch^{3/2}$ say, we have (by (3.27), since we may assume that $u^{(i)0}\in D_L$, $1\le i\le 3$) that $|\hat{e}_1{}^j|_\infty\le c(h^{3/2}+k^2+kh^{1/2})\le ch^{3/2}$, $1\le j\le 3$. In particular we note that for h sufficiently small, $U^0, \hat{U}_1{}^j$ belong to Y and

(3.28) $|e^0|_\infty\le h$, $|\hat{e}_1{}^j|_\infty\le h$, $1\le j\le 3$,

from which, by (3.10), assuming that k is sufficiently small, it follows that A_1 is invertible, i.e., that U^1 exists uniquely. Then (3.15), (3.26) and (3.28) yield (3.24). Finally, (3.24) and the estimates (4.3)-(4.5) of [2] give (3.25).□

Letting now $E^0=U^0-W^0(=0)$ and $E^1=U^1-W^1$ as in section 2, we have by (3.25), (1.6) that $\|e^1\|\le\|E^1\|+\|u^1-W^1\|\le c(k^4+h^r)$. Also, by (3.25), (iv.b) and (1.8), noting that $r\ge 2$, $1\le N\le 3$, we obtain, for h sufficiently small, that $|u^1-U^1|_\infty\le|E^1|_\infty+|u^1-W^1|_\infty\le ch^{-N/2}(k^5+kh^r)+ch^r|\ln h|^{\bar{r}}\le ch^{3/2}\le h$.

Summarizing the above estimates, we write, for later reference, the following conclusions, which are valid under the hypotheses of Proposition 3.1 and for h sufficiently small:

U^0, U^1 exist in $S_h\cap Y$;

$\|E^j\|\le c_jk(k^4+h^r)$, $j=0,1$;

(3.29) $\|E^j-E^{j-1}\|^2+k^2\|L_j{}^{1/2}(E^j-E^{j-1})\|^2+k^2\|L_j{}^{1/2}(E^j+E^{j-1})\|^2$
$+k^4\|L_j(E^j-E^{j-1})\|^2+k^4\|L_j(E^j+E^{j-1})\|^2\le c_jk^2(k^4+h^r)^2$, $j=1$;

$\|e^j\|\le c_j(k^4+h^r)$, $|e^j|_\infty\le c_jh^{3/2}\le h$, $j=0,1$.

We now turn to the calculation and the error analysis of U^j and $\hat{U}^j$, $2\le j\le 5$. We let as usual $E^j=U^j-W^j$, $e^j=u^j-U^j$, $\hat{e}^j=u^j-\hat{U}^j$.

LEMMA 3.3 *Suppose that for some* $1\le n\le J-1$, $U^n, U^{n-1}, \hat{U}^{n+1}$ *exist in* $S_h\cap V$ *and satisfy the estimates*:

For $j=n$:

(3.30) $\|E^j\|\le c_j k(k^4+h^r)$,

(3.31) $E_{j,j-1}\equiv\|E^j-E^{j-1}\|^2+k^2\|L_j^{1/2}(E^j-E^{j-1})\|^2+k^2\|L_j^{1/2}(E^j+E^{j-1})\|^2$
$+k^4\|L_j(E^j-E^{j-1})\|^2+k^4\|L_j(E^j+E^{j-1})\|^2\le c_j k^2(k^4+h^r)^2$.

For $j=n-1,n$:

(3.32) $\|e^j\|\le c_j(k^4+h^r)$,

(3.33) $|e^j|_\infty\le h$.

For $j=n$:

(3.34) $\|\hat{e}^{j+1}\|\le\hat{c}_j(k^4+h^r)$,

(3.35) $|\hat{e}^{j+1}|_\infty\le h$.

In addition, assume that (1.4) *and* (1.9) *hold, that there exists* $a>0$ *such that* $kh^{-1}\le a$ *and that* k,h *are sufficiently small. Suppose that the point* (q_1,q_2) *belongs in the stability region* $\bar{\mathcal{R}}$, cf. section 1, *and that the stability hypothesis* (2.24) *of* [3] (*of the form* kh^{-1} *small*) *holds if* $(q_1,q_2)\in\partial R_3\cup B$ *in* Fig. 1 *of* [3]. *Then* U^{n+1}, *the solution of* (1.13), *exists uniquely. Moreover, there exists a constant* $c_{n+1}>0$ *such that* (3.30)-(3.33) *hold for* $j=n+1$.

Proof. (3.35) implies that $\hat{U}^{n+1}\in V$ for h sufficiently small. Consequently, (2.4) in the proof of Lemma 2.2 is valid and yields the invertibility of $\hat{A}_{n+1}$ on S_h, i.e., the existence-uniqueness of U^{n+1}. Moreover, our hypotheses imply that (2.89) holds for $l=m=n$. Inserting in (2.89) the estimates (3.31) for $j=n$, (3.32) for $j=n,n-1$, (3.33) for $j=n,n-1$ and (3.34) for $j=n$, it is not hard to see that there exists a constant $c>0$ and, for any $\varepsilon>0$, a constant $c(\varepsilon)>0$ ($c,c(\varepsilon)$ depend on c_n,c_{n-1}), such that, with $\eta^{(1)}_j$ defined by (2.19),

(3.36) $\eta^{(1)}_{n+1}\le c(\varepsilon)k^2(k^4+h^r)^2+(ck+\varepsilon)\,E_{n+1,n}$.

By our assumptions on q_l,p_l that concern the accuracy and stability of the scheme, we obtain, mutatis mutandis of course, but essentially exactly as in the proof of Theorem 2.1 of [3], that, for ε,k sufficiently small, we may hide the terms of $E_{n+1,n}$ in the appropriate terms of $\eta^{(1)}_{n+1}$ and bound below the resulting left-hand side of (3.36) by a constant times $E_{n+1,n}$, thus obtaining

(3.37) $$E_{n+1,n}\le c_{n+1}k^2(k^4+h^r)^2,$$

i.e., (3.31) for $j=n+1$. This is the key estimate from which the others follow easily. Indeed, since $\|E^{n+1}\|\le\|E^{n+1}-E^n\|+\|E^n\|$, (3.37) and (3.30, $j=n$) give (3.30, $j=n+1$). Since $\|e^{n+1}\|\le\|E^{n+1}\|+\|u^{n+1}-W^{n+1}\|$, (3.32, $j=n+1$), follows. Finally, since $|e^{n+1}|_\infty\le|E^{n+1}|_\infty+|u^{n+1}-W^{n+1}|_\infty\le ch^{-N/2}c_{n+1}(k^5+kh^r)+ch^{3/2}\le cc_{n+1}h^{3/2}$, it follows, for h so small that $cc_{n+1}h^{1/2}<1$, that (3.33) holds for $j=n+1$ as well.□

We stress again that we do not intend to use this stability lemma for estimating the error over a large number of steps, since such use would require a (nonexistent) a priori bound for the constants c_n, independent of h,k and n. However, since (3.29) implies (3.30), $j=0,1$, (3.31), $j=1$ and (3.32) and (3.33) for $j=0,1$, and since $U^0, U^1 \in S_h \cap V$, there follows that we may use Lemma 3.3 in an inductive fashion for a few steps, provided that we furnish for each j a $\hat{U}^{j+1}$ that satisfies estimates of the form (3.34) and (3.35). To this end, define first

$$\hat{U}^2=8U^1-7U^0-6kPu^{(1)o}-2k^2Pu^{(2)o} \tag{3.38.2}$$

Then, since by (3.1), $\hat{U}^2-u^2=8(U^1-W^1)-7(W^0-u^0)+8(W^1-u^1)+8u^1-7u^0-6ku^{(1)o}-2k^2u^{(2)o}-u^2-6k(P-I)u^{(1)o}-2k^2(P-I)u^{(2)o}$, it follows, by (3.29), $kh^{-1}\leq a$ and techniques similar to those that led to (3.26), that (3.34) holds for $j=1$. In addition, estimating in L^∞ and using (3.29), (1.8), (3.27), $kh^{-1}\leq a$, we see that (3.35) holds for $j=1$. Applying Lemma 3.3 for $j=1$ now, we see (under its additional hypotheses) that its conclusion is valid for $j=2$. Let then

$$\hat{U}^3=(9/2)U^2-9U^1+(11/2)U^0+3kPu^{(1)o}. \tag{3.38.3}$$

It may be seen, as above, that $\hat{U}^3$ satisfies (3.34) and (3.35) for $j=2$. Hence the conclusion of Lemma 3.3 for $n=2$. Continue by setting

$$\hat{U}^4=4U^3-6U^2+4U^1-U^0, \tag{3.38.4}$$

and deduce that the conclusions of Lemma 3.3 hold for $n=3$ as well. Finally defining

$$\hat{U}^5=4U^4-6U^3+4U^2-U^1, \tag{3.38.5}$$

we see that the conclusions of Lemma 3.3 are valid for $n=4$. Obviously, we could go on for a (small) number of additional steps, defining $\hat{U}^{n+1}=4U^n-6U^{n-1}+4U^{n-2}-U^{n-4}$ and obtaining each time the results of Lemma 3.3. Of course, this cannot continue for long since we have no control of the growth with n of the constants $c_{n+1}, \hat{c}_{n+1}$ (equivalently, since we have no guarantee that h,k will not shrink to zero as n grows).

S5. APPENDIX

In this section we collect some remarks concerning the validity of hypotheses (1.4), (1.9) and (iv.c). We shall assume, since $S_h \subset W^{1,\infty}$ and $1 \le N \le 3$, that the following general inverse property holds on S_h, cf. [7]:

$$(5.1) \quad \|\varphi\|_{l,p} \le c h^{m-l-N(q^{-1}-p^{-1})} \|\varphi\|_{m,q}, \quad \forall \varphi \in S_h,$$

for $0 \le m \le l \le 1$, $1 \le q \le p \le \infty$. (5.1) is true in general for a quasi-uniform triagulation.

To justify (1.4) now, we assume that the L^2 projection operator onto S_h is stable in L^∞; this is also valid in general for quasi-uniform triangulations and it is proved in [13]. Specifically, assume that there is a constant c such that for all $v \in L^\infty(\Omega)$ we have

$$(5.2) \qquad |Pv|_\infty \le c|v|_\infty .$$

The desired inequality (1.4) follows from (5.2) and (5.1) with $l=1$, $m=0$, $p=q=\infty$. For the stability of P directly in the $W^{1,\infty}$ norm without inverse assumptions, cf. [8].

We shall justify (1.9) in the case of the standard Galerkin method under some additional hypotheses that will be specified below. First, augmenting (1.8), we assume that the elliptic projection $P_1=P_1(t)$ satisfies, for $1 \le N \le 3$, $r \ge 2$ and $v \in W^{r,\infty} \cap D_L$, the estimate

$$(5.3) \qquad |v-P_1v|_\infty + h\|v-P_1v\|_{1,\infty} \le c(v)h^r|\log h|^{\bar r},$$

where $\bar r$ is as in (1.8). For the validity of (5.3) for the standard Galerkin method, cf., e.g., [17], [20].

From (5.3) it follows that (1.9) holds for $j=0$, $1 \le N \le 3$, $r \ge 2$. Thus, we examine henceforth the case $j=1$. Since for any $\chi \in S_h$ we have, using (5.1), that

$$\|W^{(1)}\|_{1,\infty} \le \|W^{(1)}-u^{(1)}\|_{1,\infty} + \|u^{(1)}\|_{1,\infty} \le \|W^{(1)}-\chi\|_{1,\infty} + \|\chi-u^{(1)}\|_{1,\infty} + c$$
$$\le ch^{-(1+N/2)}(\|W^{(1)}-u^{(1)}\| + \|\chi-u^{(1)}\|) + \|\chi-u^{(1)}\|_{1,\infty} + c,$$

setting $\chi=P_1u^{(1)}$ and using (1.6) and (5.3), we conclude that if $r \ge 1+N/2$,

$$\|W^{(1)}\|_{1,\infty} \le ch^{r-1-N/2} + ch^{r-1}|\log h|^{\bar r} + c \le c .$$

Hence (1.9), $j=1$, holds for $r \ge 2$ if $N=1$ or 2, but this proof requires that $r \ge 3$ if $N=3$. Therefore, we now concentrate upon proving (1.9) for $j=1$ in the case $r=2$, $N=3$. To this end we make two additional assumptions: first we suppose that given a function $w=w(x)$, sufficiently smooth on $\bar\Omega$, there exists a constant $c(w)$ such that the following superapproximation property, cf. (6.11) of [18] and its references, holds:

$$(5.4) \qquad \inf_{\varphi \in S_h} \|w\chi-\varphi\|_1 \le c(w)\|\chi\|, \quad \forall \chi \in S_h .$$

We also assume that the coefficients of the principal part of the operator $L=L(x,t,u)$ in (1.1) are of the form

$$(5.5) \quad a_{ij}(x,t,u)=A(x,t,u)\tilde a_{ij}(x), \quad 1 \le i,j \le N, \quad (x,t,u) \in Q_\delta,$$

where $A(x,t,u)\geq a>0$ in Q_δ and $\tilde{a}_{ij}(x)$ is a symmetric uniformly positive definite matrix on $\bar{\Omega}$. If u is the solution of (1.1), in addition to the bilinear form $a(t,u)(.,.)$ introduced in section 1, we consider for $v,w\in H^1$

$$(5.6)\quad a^{(1)}(t,u,u_t)(v,w)=\int_\Omega\Big(\Sigma^3_{i,j=1}\,(D_t[a_{ij}(x,t,u)]\partial_i v\partial_j w)+D_t[a_0(x,t,u)]vw\Big)dx.$$

Recall that the elliptic projection $P_1(t)v$ satisfies

(5.7) $a(t,u)(P_1(t)v,\chi)=a(t,u)(v,\chi)\ \ \forall\chi\in S_h,\ \ t\in[0,t^*]$.

Setting $v=u$ in the above and differentiating with respect to t, we obtain for $t\in[0,t^*]$, $\forall\chi\in S_h$,

$a^{(1)}(t,u,u_t)(W,\chi)+a(t,u)(W^{(1)},\chi)=a^{(1)}(t,u,u_t)(u,\chi)+a(t,u)(u^{(1)},\chi)$

Since (5.7) with $v=u^{(1)}$ yields $a(t,u)(u^{(1)},\chi)=a(t,u)(P_1u^{(1)},\chi)$, we conclude for $t\in[0,t^*]$, $\forall\chi\in S_h$:

$$(5.8)\quad a(t,u)(W^{(1)}-P_1(t)u^{(1)},\chi)=a^{(1)}(t,u,u_t)(u-W,\chi).$$

Using (5.5) and (5.6), we see that $a^{(1)}(t,u,u_t)(u-W,\chi)$

$=\Sigma^3_{i,j=1}\ (D_t[A(\ ,t,u)]\tilde{a}_{ij}\ \partial_i(u-W),\partial_j\chi)+(D_t[a_0(.,t,u)](u-W),\chi)$.

Hence defining $w=w(x,t)=D_t[A(x,t,u)]/A(x,t,u)$ - by our assumptions we have that w is smooth on Q_δ - we conclude by the above identities that for any $\chi,\psi\in S_h$, $t\in[0,t^*]$,

$$(5.9)\quad a^{(1)}(t,u,u_t)(u-W,\chi)=\Sigma^3_{i,j=1}\,(A\,\tilde{a}_{ij}\partial_i(u-W),\ w\partial_j\chi)$$
$$+((D_ta_0-wa_0)(u-W),\chi)+(wa_0(u-W),\chi)$$
$$=a(t,u)(u-W,w\chi)-\Sigma^3_{i,j=1}(A\tilde{a}_{ij}\partial_i(u-W),(\partial_j w)\chi)+([D_ta_0-wa_0](u-W),\chi)$$
$$=a(t,u)(u-W,w\chi-\psi)-\Sigma^3_{i,j=1}(A\tilde{a}_{ij}\partial_i(u-W),(\partial_j w)\chi)+([D_ta_0-wa_0](u-W),\chi).$$

Hence,

$$|a^{(1)}(t,u,u_t)(u-W,\chi)|\leq c\|u-W\|_1(\|w\chi-\psi\|_1+\|\chi\|)\ \ \forall\chi,\psi\in S_h,$$

which implies, by (5.4) and (5.8), that

$$(5.10)\quad |a(t,u)(W^{(1)}-P_1(t)u^{(1)},\chi)|\leq c\|u-W\|_1\ \|\chi\|\ \ \forall\chi\in S_h,\ \ t\in[0,t^*].$$

Our intention is to use as χ a type of discrete Green's function on Ω bounded in L^2 independently of h in order to produce the maximum norm of $W^{(1)}-P_1u^{(1)}$ in the left-hand side of (5.10). To this end we may assume that for $r=2$, $N=3$, given $t\in[0,t^*]$, $x\in\Omega$, there exists $G_h\in S_h$ such that for each $\varphi\in S_h$

$$(5.11)\qquad a(t,u)(\varphi,G_h)=\varphi(x),$$

such that for some c independent of x,t,h:

$$(5.12)\qquad \|G_h\|\leq c.$$

For a justification of (5.11), (5.12) we refer the reader to [15], [16], where it is proved that such a G_h exists and satisfies $\|G_h - G\| \le ch^{1/2}$, where $G = G^x$ is the Green's function for our elliptic operator (for a fixed t) with singularity at $x \in \Omega$. Since in three dimensions, $|G^x(y)| \le c|y-x|^{-1}$, we conclude that G^x is uniformly bounded in $L^2(\Omega)$; (5.11) follows.

The rest of the proof is now straightforward. Putting $\chi = G_h$ in (5.10), we have, using (5.11) and (5.12), that $|W^{(1)} - P_1(t)u^{(1)}|_\infty \le c\|u-W\|_1 \le ch$. This, in conjunction with (5.1), the triangle inequality and (5.3) with $v = u^{(1)}$, $r=2$, yields (1.9) for $j=1$ in the case $r=2$, $N=3$ as well.

We also mention the following simplification of the proof of (1.9) for $j=1$, $r=2$, $N=3$ (in the case of the standard Galerkin method) which was pointed out to us by the referee of the first version of this paper: A rearrangement of the terms in the right-hand side of (5.9) yields the identity

$$(5.13)\quad a^{(1)}(t,u,u_t)(u-W,\chi) = a(t,u)(w(u-W),\chi)$$
$$-\sum_{i,j=1}^{3} (A\tilde{a}_{ij}(u-W)\partial_i w, \partial_j \chi) + ([D_t a_0 - w a_0](u-W),\chi).$$

Extending the domain of definition of the elliptic projection operator onto S_h to all of $\mathring{H}^1$ by defining $P_1(t)v$ for $v \in \mathring{H}^1$ by (5.7) and denoting the extension by $P_1(t)$ as well, we have in (5.13) that $a(t,u)(w(u-W),\chi) = a(t,u)(P_1(w(u-W)),\chi)$. Thus, (5.13) and (5.8) yield

$$(5.14)\qquad W^{(1)} = P_1 u^{(1)} + P_1(w(u-W)) + \eta,$$

where $\eta \in S_h$ satisfies, for every $\chi \in S_h$,

$$(5.15)\quad a(t,u)(\eta,\chi) = -\sum_{i,j=1}^{3} (A\tilde{a}_{ij}(u-W)\partial_i w, \partial_j \chi) + ([D_t a_0 - w a_0](u-W),\chi).$$

Assuming now that the elliptic projection is quasi-optimal in the L^∞ norm modulo a logarithmic factor (cf. [20]) and using (5.1) and (5.3) for $r=2$, we have, since w is smooth, for any $\varepsilon > 0$ that there exists $c_\varepsilon > 0$ such that

$$(5.16)\quad \|P_1(w(u-W))\|_{1,\infty} \le ch^{-1}|P_1(w(u-W))|_\infty \le ch^{-1}|\log h|\,|w(u-W)|_\infty \le c_\varepsilon h^{1-\varepsilon}.$$

On the other hand, putting $\chi = \eta$ in (5.15) yields $\|\eta\|_1 \le c\|u-W\|$. Hence, (5.1) and (1.5) give $\|\eta\|_{1,\infty} \le ch^{1/2}$. We conclude by (5.14), (5.16) that $\|W^{(1)} - P_1 u^{(1)}\|_{1,\infty} \le ch^{1/2}$, which gives (1.9) for $j=1$, $N=3$, $r=2$, since $P_1 u^{(1)}$ is bounded in $W^{1,\infty}$ by (5.3).

To justify (iv.c), observe that for the standard Galerkin method $\|L_0^{1/2}\cdot\|$ is comparable to $\|\cdot\|_1$ on S_h. Therefore (iv.c) holds with $\gamma(h)=1$ if $N=1$ (Sobolev's inequality), while for $N=2$ one may take $\gamma(h) = |\log h|^{1/2}$, cf. [19]. For $N=3$ one may readily prove (iv.c) with $\gamma(h) = h^{-1/2}$, extending in a straightforward way the argument of Lemma 1.1, p.274 of [19] to three dimensions. As pointed out to us by the referee, one may simplify this proof, following a suggestion by V. Thomée, cf. [20], as follows. For $N=3$, using the L^∞-L^6 inverse inequality in (5.1), we have $|\phi|_\infty \le ch^{-1/2}\|\phi\|_{0,6}$, $\phi \in S_h$. The desired estimate (iv.c) with $\gamma(h) = h^{-1/2}$ now follows using the continuous imbedding (Sobolev's theorem, $N=3$) of H^1 into L^6.